Flat Metric Minimization with Applications in Generative Modeling

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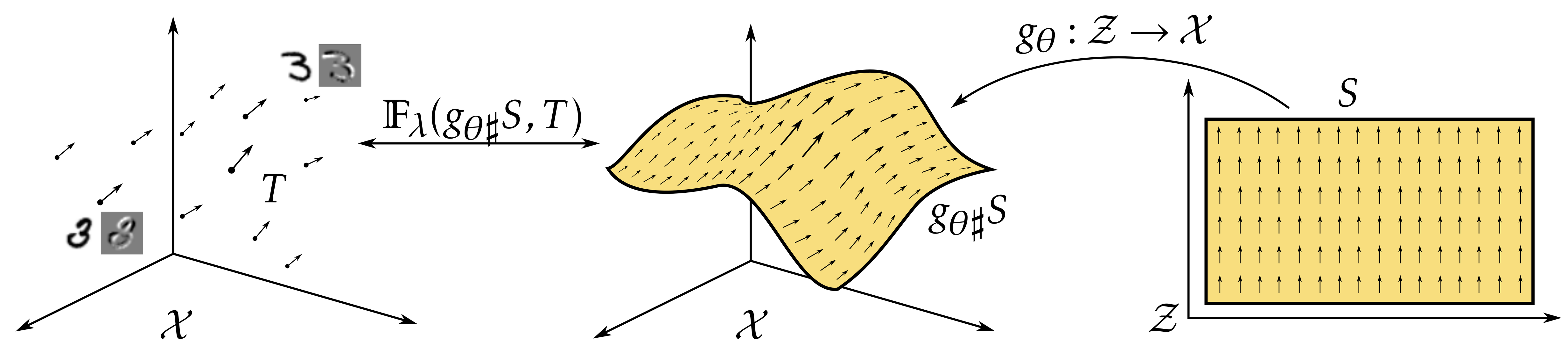
Representing Data with Normal Currents

Contribution: We propose to view (partially) oriented data as a k-current.

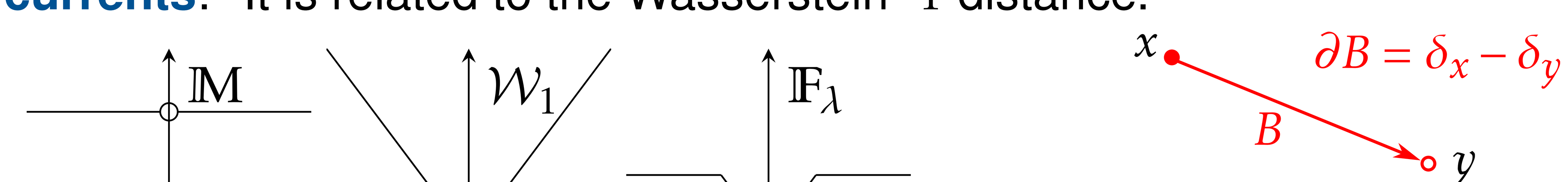


Intuitively, k-**currents** form a linear space that includes k-dimensional oriented manifolds as elements. The vector space of **normal currents** $\mathbf{N}_{k,\mathcal{X}}(\mathbf{R}^d)$ consists of currents T with finite volume and finite volume of their boundary: $\mathbb{M}(T) + \mathbb{M}(\partial T) < \infty$.

The Flat Metric

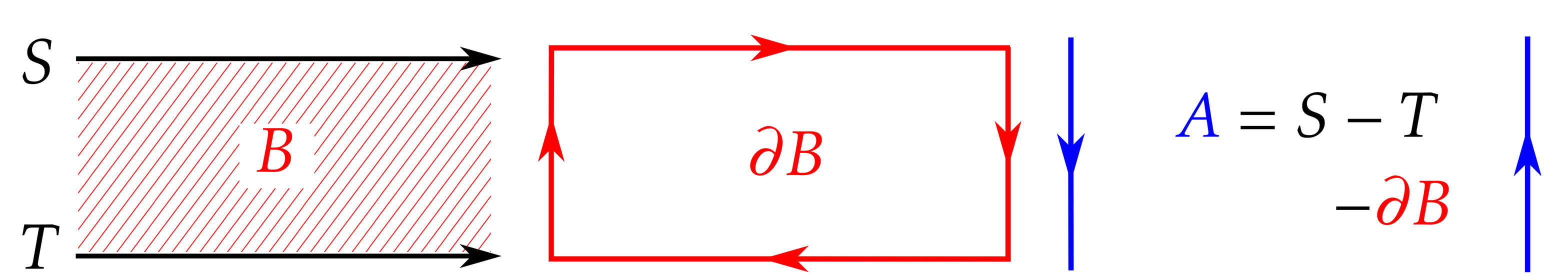
$$\mathbb{F}_\lambda(S,T) = \min_{S-T=\partial B + A} \mathbb{M}(B) + \lambda \mathbb{M}(A) = \sup_{\substack{\|\omega\|^* \leq \lambda \\ \|d\omega\|^* \leq 1}} S(\omega) - T(\omega)$$

For 0-currents: It is related to the Wasserstein-1 distance.



$$\mathbb{F}_\lambda(\delta_x, \delta_y) = \min\{\lambda, \|x-y\|\}$$

The intuition for 1-currents:



Theoretical Results

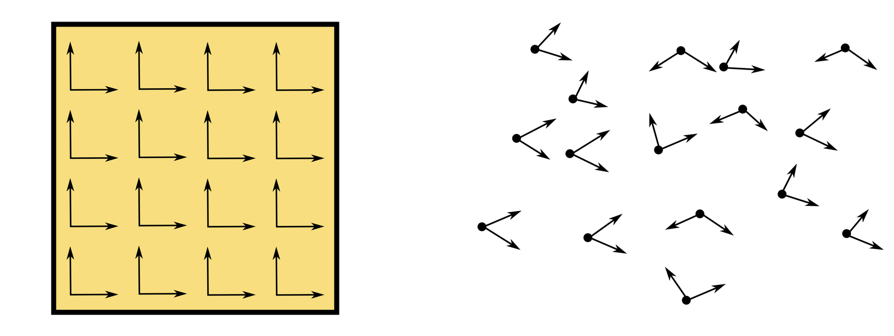
Federer & Fleming 1960. The flat metric metrizes the weak* convergence on normal currents with uniformly bounded mass and boundary mass:

$$\mathbb{F}_\lambda(T, T_i) \to 0 \text{ if and only if } T_i \overset{*}{\rightharpoonup} T, \text{ i.e., } T_i(w) \to T(w), \text{ for all } \omega \in C_c^\infty(\mathbf{R}^d; \mathbf{\Lambda}^k \mathbf{R}^d).$$

Proposition. Let $S \in \mathbf{N}_{k,\mathcal{Z}}(\mathbf{R}^l)$, $T \in \mathbf{N}_{k,\mathcal{X}}(\mathbf{R}^d)$ be normal currents. Assume $g_\theta : \mathcal{Z} \to \mathcal{X}$ is smooth in z with uniformly bounded derivative and locally Lipschitz in θ. Then, the map $\theta \mapsto \mathbb{F}_\lambda(g_{\theta\sharp}S, T)$ is Lipschitz continuous on any compact parameter set Θ.

Presented at the International Conference on Machine Learning (ICML), Los Angeles, 2019.

FlatGAN: Learning Equivariant Representations

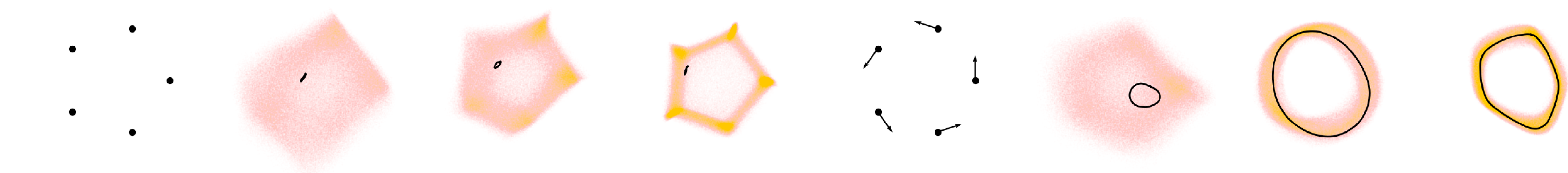


$S = \mu \wedge (e_1 \wedge \ldots \wedge e_k)$ $T = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \wedge T_i$

$$\min_{\theta \in \Theta} \left\{ \mathbb{F}_\lambda(g_{\theta\sharp}S, T) = \sup_{\substack{\|\omega\|^* \leq \lambda \\ \|d\omega\|^* \leq 1}} -\frac{1}{N}\sum_{i=1}^N \langle \omega(x_i), T_i \rangle + \mathbb{E}_{z \sim \mu}[\langle \omega \circ g_\theta, (\nabla_z g_\theta \cdot e_1) \wedge \ldots \wedge (\nabla_z g_\theta \cdot e_k) \rangle] \right\}.$$

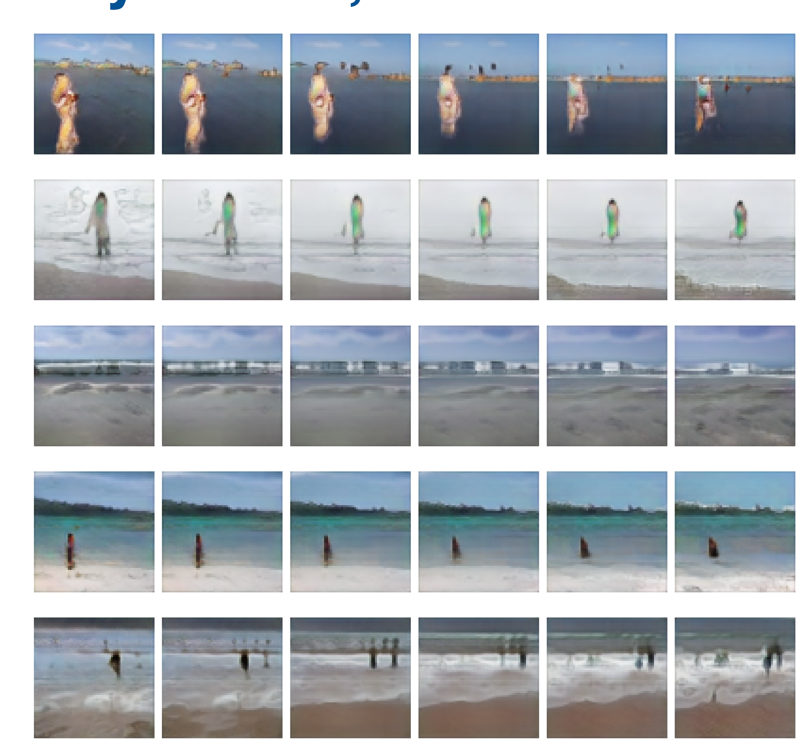
Solving the above optimization problem yields a generator g_θ which behaves equivariantly to the specified tangent vectors.

Illustration on a simple dataset in 2D:



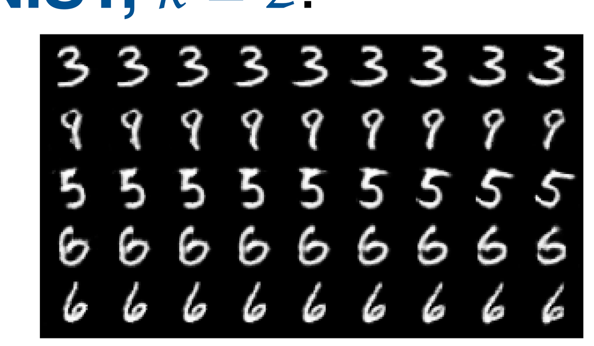
$T \in \mathbf{N}_{0,\mathcal{X}}(\mathbf{R}^d)$ it. 500 it. 1000 it. 2000 $T \in \mathbf{N}_{1,\mathcal{X}}(\mathbf{R}^d)$ it. 500 it. 1000 it. 2000

tinyvideos, $k = 1$:

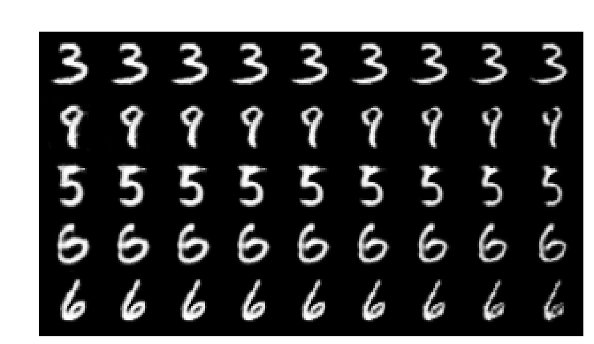


varying z_1 (time)

MNIST, $k = 2$:



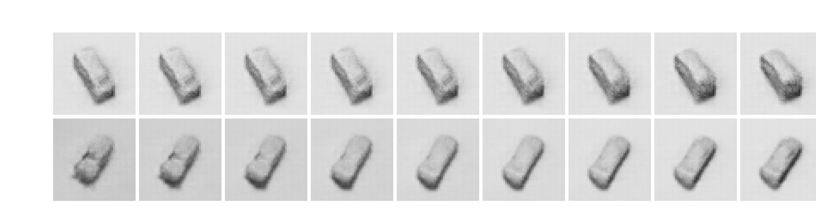
varying z_1 (rotation)



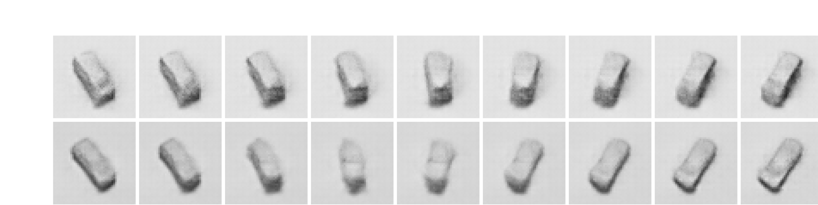
varying z_2 (stroke width)

smallNORB, $k = 3$:

varying lighting (z_1)



varying elevation (z_2)



varying azimuth (z_3)

Geometric Measure Theory Cheat Sheet & References

- k**-vectors and** k**-covectors.** $\mathbf{\Lambda}_k\mathbf{R}^d$ is a vector space in which some of the elements describe oriented, k-dimensional planes in $\mathbf{R}^d$. These are called **simple** k-vectors: $v_1 \wedge \ldots \wedge v_k$. The dual space ($k$-covectors) is $\mathbf{\Lambda}^k\mathbf{R}^d$.
- If v and w are simple, then we have $\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle = \det(V^\top W)$.
- A **differential form** is a k-covector field $\omega : \mathbf{R}^d \to \mathbf{\Lambda}^k\mathbf{R}^d$. k**-currents** are the dual space of smooth, compact k-forms.
- $\|v\| = \sup_{\|w\|^* \leq 1} \langle v, w \rangle$. Area of the k-dim. parallelotope spanned by the $\{v_i\}$ if $v = v_1 \wedge \ldots \wedge v_k$.
- The **mass** $\mathbb{M}(T) = \sup_{\|\omega\|^* \leq 1} T(w)$ is the k-dimensional *volume* of the k-current T.
- **Boundary:** $\partial T(\omega) = T(d\omega)$, where d is the exterior derivative (in $\mathbf{R}^3$: grad → curl → div)
- **Orientation:** Continuous k-vector map $\tau_\mathcal{M} : \mathcal{M} \to \mathbf{\Lambda}_k\mathbf{R}^d$, $\tau_\mathcal{M}(z)$ is simple with unit norm, spanning $T_z\mathcal{M}$ for all $z \in \mathcal{M}$.
- **Stokes' theorem:** $\int_\mathcal{M} \langle d\omega, \tau_\mathcal{M} \rangle \mathrm{d}\mathcal{H}^k = \int_{\partial\mathcal{M}} \langle \omega, \tau_{\partial\mathcal{M}} \rangle \mathrm{d}\mathcal{H}^{k-1}$, it follows that $\partial[\![\mathcal{M}]\!] = [\![\partial\mathcal{M}]\!]$.
- **Pullback:** $\langle g^\sharp\omega, v_1 \wedge \ldots \wedge v_k \rangle = \langle \omega \circ g, \nabla g \cdot v_1 \wedge \ldots \wedge \nabla g \cdot v_k \rangle$, **pushforward:** $g_\sharp T(\omega) = T(g^\sharp\omega)$.

[1] H. Federer and W. H Fleming. Normal and integral currents. *Annals of Mathematics*, pages 458–520, 1960.

[2] H. Federer. *Geometric Measure Theory*. Springer, 1969.

[3] F. Morgan. *Geometric Measure Theory: A Beginner's Guide*. Academic Press, 5th edition, 2016.